Vector algebra

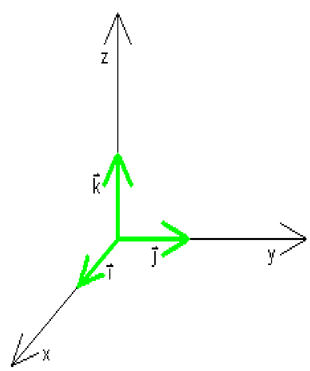
In the sequel we consider only the three dimensional Euclidian vector spaces, denoted by V_3.

Based on the usual notations in $\mathbb{R}^3$, a point P_0 can be written in Cartesian coordinate form as $P_0 = (x_0, y_0, z_0)$.

We will denote by $\overrightarrow{OP_0}$ the oriented line sequence, the position vector of the point P_0, and for which we introduce the similar coordinates $\overrightarrow{OP_0} = \langle x_0, y_0, z_0\rangle$.

We will have for two different points $P_1 = (x_1, y_1, z_1)$. and $P_2 =(x_2, y_2, z_2)$ in $\mathbb{R}^3$, the oriented line sequence will be denoted by $\overrightarrow{P_1P_2}$, and we will use the coordinate form

$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$, as obviously $\overrightarrow{P_1P_2} = \overrightarrow{OP_2}$ -$\overrightarrow{OP_1}$. The class of conguence $\left\{\overrightarrow{P_1P_2} = \langle a, b, c\rangle \mid P_1, P_2 \in \mathbb{R}^3\right\}$ is by definition the vector $\overrightarrow{v} = \langle a, b, c\rangle \in V_3$. We mention 3 special vectors denoted $\overrightarrow{i} = \langle 1, 0, 0\rangle$, $\overrightarrow{j} = \langle 0, 1, 0\rangle$, $\overrightarrow{k} = \langle 0, 0, 1\rangle$, the unit vectors of the 3 coordinate axis, named coordinate vectors.



Basic vector operations

We have $\overrightarrow{v} = \langle a, b, c\rangle = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$.

Vector multiplied by a scalar

Let us take the field $(\mathbb{R}, +, \cdot)$ and the Abelian group $(V_3, +)$. We define an external operation type $\mathbb{R}\times V_3 \longrightarrow V_3$, i.e. the operation of multiplying the vector $\overrightarrow{v} = \langle a, b, c\rangle \in V_3$ by the scalar λ, denoted by $\lambda\overrightarrow{v} = \langle \lambda a, \lambda b, \lambda c\rangle \in V_3$.

Properties

The vector $\lambda\overrightarrow{v}$ will be parallel with $\overrightarrow{v}$, except for $\lambda = 0$.

Example. For $\overrightarrow{v} = \langle 2, -1, 3\rangle$ and $\lambda = 5$ will furnish the vector $5\overrightarrow{v} = \langle 10, -5, 15\rangle$

We will be able to check this using e.g their vector product, see below.

Further properties:
$\lambda(\overrightarrow{v_1}+\overrightarrow{v_2})=\lambda\overrightarrow{v_1}+\lambda\overrightarrow{v_2}$
$(\lambda_1+\lambda_2)\overrightarrow{v}=\lambda_1\overrightarrow{v}+\lambda_2\overrightarrow{v}$
$\lambda_1(\lambda_2\overrightarrow{v})=(\lambda_1\lambda_2)\overrightarrow{v}$ and
$1.\overrightarrow{v}=\overrightarrow{v}$, where *1* is the unit in $\mathbb{R}$.

Scalar product of two vectors (dot product)

We define the external operation type $V_3 \times V_3 \longrightarrow \mathbb{R}$ in the following way:

Given any two vectors $\overrightarrow{v_1}=\langle a_1,b_1,c_1\rangle$, $\overrightarrow{v_2}=\langle a_2,b_2,c_2\rangle\in V_3$,their scalar product, (named sometimes dot product) is: $\overrightarrow{v_1}\cdot\overrightarrow{v_2}=a_1a_2+b_1b_2+c_1c_2\in\mathbb{R}$.

Properties
$\overrightarrow{v_1}\cdot\overrightarrow{v_2}=\overrightarrow{v_2}\cdot\overrightarrow{v_1}$ commutativity
$\overrightarrow{v_1}\cdot(\overrightarrow{v_2}+\overrightarrow{v_3})=\overrightarrow{v_1}\cdot\overrightarrow{v_2}+\overrightarrow{v_1}\cdot\overrightarrow{v_3}$ linearity
$\overrightarrow{v}\cdot\overrightarrow{v}\geq 0$, the last one is used to introduce $|\overrightarrow{v}|=\sqrt{\overrightarrow{v}\cdot\overrightarrow{v}}$, named the length of the vector $\overrightarrow{v}$ (norm).

The unit vector $\overrightarrow{u^0}$ of the vector $\overrightarrow{u}$ is $\overrightarrow{u^0}=\frac{\overrightarrow{u}}{|\overrightarrow{u}|}$, e.g. $\langle 3,4,12\rangle^0=\left\langle\frac{3}{13},\frac{4}{13},\frac{12}{13}\right\rangle$.

The scalar product of two vectors $\overrightarrow{v_1}\cdot\overrightarrow{v_2}$ has an other interpretation:
$\overrightarrow{v_1}\cdot\overrightarrow{v_2}=|\overrightarrow{v_1}|\,|\overrightarrow{v_2}|\cos\varphi$, where φ denotes the angle of the two vectors.

Applications

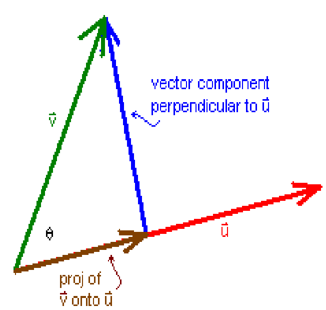
We deduce: $\cos\varphi=\frac{\overrightarrow{v_1}\cdot\overrightarrow{v_2}}{|\overrightarrow{v_1}||\overrightarrow{v_2}|}$, and we get an equavalent condition for the perpendicularity of two vectors, i.e. the nonzero vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are perpendicular iff $\overrightarrow{v_1}\cdot\overrightarrow{v_2}=0$.(iff stands here for if and only if).

Vector projection

In order to define the projection of a vector $\overrightarrow{v}$ onto vector $\overrightarrow{u}$ we need first to get the lenght of the projection. If we check the figure below, we observe that
$\overrightarrow{v}\cdot\overrightarrow{u^0}$

is exacly what we need, i.e. $\overrightarrow{v}\cdot\overrightarrow{u^0}=|\overrightarrow{v}|\cos\varphi$.

The projection we look for is: $pr_{\overrightarrow{u}}\overrightarrow{v}=\left(\overrightarrow{v}\cdot\overrightarrow{u^0}\right)\overrightarrow{u^0}=\frac{(\overrightarrow{v}\cdot\overrightarrow{u})\overrightarrow{u}}{|\overrightarrow{u}|^2}$.



Vector product of two vectors (cross product)

We define the internal operation type $V_3 \times V_3 \longrightarrow V_3$ in the following way:

Given any two vectors $\overrightarrow{v_1} = \langle a_1, b_1, c_1\rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2\rangle \in V_3$,their vector product, (named sometimes cross product) is:

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = \langle b_1c_2 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1\rangle \in V_3.$$

This definition can be easier memorised if taking the following formal definition (using a formal determinant):

$\overrightarrow{v_1} \times \overrightarrow{v_2} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$, where $\overrightarrow{i} = \langle 1, 0, 0\rangle$, $\overrightarrow{j} = \langle 0, 1, 0\rangle$, $\overrightarrow{k} = \langle 0, 0, 1\rangle$ are the coordinate vectors already mentioned.

Properties

$\overrightarrow{v_1} \times \overrightarrow{v_2} \perp \overrightarrow{v_1}, \overrightarrow{v_1} \times \overrightarrow{v_2} \perp \overrightarrow{v_2}$.

$\overrightarrow{v_1} \times \overrightarrow{v_2} = -\overrightarrow{v_2} \times \overrightarrow{v_1}$ anti-commutativity

$\overrightarrow{v_1} \times (\overrightarrow{v_2} + \overrightarrow{v_3}) = \overrightarrow{v_1} \times \overrightarrow{v_2} + \overrightarrow{v_1} \times \overrightarrow{v_3}$ and

$(\overrightarrow{v_1} + \overrightarrow{v_2}) \times \overrightarrow{v_3} = \overrightarrow{v_1} \times \overrightarrow{v_3} + \overrightarrow{v_2} \times \overrightarrow{v_3}$ linearity

We can express the vector product of two vectors $\overrightarrow{v_1} = \langle a_1, b_1, c_1\rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2\rangle \in V_3$ as a vector perpendicular the same time on both vectors, and with the lenght: $|\overrightarrow{v_1} \times \overrightarrow{v_2}| = |\overrightarrow{v_1}|\,|\overrightarrow{v_2}| \sin\varphi$, oriented acording to the "right hand" rule, i.e. the vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ and $\overrightarrow{v_1} \times \overrightarrow{v_2}$ are in the similar position to the first 3 fingers on anybody's rigth hand.

Geometric application.

$|\overrightarrow{v_1} \times \overrightarrow{v_2}|$ = the area of the parallelogramm spanned by the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$.

Mixed product of three vectors

We define the external ternary operation type $V_3 \times V_3 \times V_3 \longrightarrow \mathbb{R}$ in the following way:

Given any three vectors $\overrightarrow{v_1} = \langle a_1, b_1, c_1\rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2\rangle$, $\overrightarrow{v_3} = \langle a_3, b_3, c_3\rangle \in V_3$,their mixed product, (named sometimes triple product) is:

$(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) \longmapsto (\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3}$.

For easier memorization, we can use the determinant:

$(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Properties

$(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} = -(\overrightarrow{v_1} \times \overrightarrow{v_3}) \cdot \overrightarrow{v_2} =$

$= (\overrightarrow{v_2} \times \overrightarrow{v_3}) \cdot \overrightarrow{v_1} = -(\overrightarrow{v_2} \times \overrightarrow{v_1}) \cdot \overrightarrow{v_3} =$

$= (\overrightarrow{v_3} \times \overrightarrow{v_1}) \cdot \overrightarrow{v_2} = -(\overrightarrow{v_3} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_1}$

Geometric interpretation

$|(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3}|$ = volume of the parallellipeped spanned by the three vectors, $\overrightarrow{v_1} = \langle a_1, b_1, c_1\rangle$, $\overrightarrow{v_2} = \langle a_2, b_2, c_2\rangle$, $\overrightarrow{v_3} = \langle a_3, b_3, c_3\rangle \in V_3$. The sign of the mixed product expresses if the third vector is in the same "half-space" as the vector product of the first two, i.e. they form a right hand system ($(\overrightarrow{v_1} \times \overrightarrow{v_2}) \cdot \overrightarrow{v_3} \geqq 0$) or not.

The Serret-Frénét frame

Let us take the following vector-function (vector valued real function):

$\overrightarrow{r}:\mathbb{R}\longrightarrow V_3$ given by $\overrightarrow{r}(t)=\langle x(t),y(t),z(t)\rangle$, where $x(t),y(t),z(t)$ are the coordinate functions type $\mathbb{R}\longrightarrow\mathbb{R}$. This is the vector which is characterizing the moving point $P\left(x(t),y(t),z(t)\right)$, given with the same coordinate functions in a parametric way. The point P will move along the parametric curve in the space.

It is known from mechanics that the velocity of the point is given by the

$\overrightarrow{v}=\frac{\partial\overrightarrow{r}}{\partial t}=\left\langle \dot{x}(t),\dot{y}(t),\dot{z}(t)\right\rangle$, where $\dot{x}(t)=\frac{\partial x(t)}{\partial t}$, $\dot{y}(t)=\frac{\partial y(t)}{\partial t}$, and $\dot{z}(t)=\frac{\partial z(t)}{\partial t}$, while its acceleration $\overrightarrow{a}=\frac{\partial^2\overrightarrow{r}}{(\partial t)^2}=\left\langle \ddot{x}(t),\ddot{y}(t),\ddot{z}(t)\right\rangle$, and similarly we have $\ddot{x}(t)=\frac{\partial^2 x(t)}{(\partial t)^2}$, $\ddot{y}(t)=\frac{\partial^2 y(t)}{(\partial t)^2}$, and $\ddot{z}(t)=\frac{\partial^2 z(t)}{(\partial t)^2}$.

Example: For $\overrightarrow{r}(t)=\langle\cos t,\sin t,e^{-t}\rangle$, we have $\overrightarrow{v}=\langle-\sin t,\cos t,-e^{-t}\rangle$ and $\overrightarrow{a}=\langle-\cos t,-\sin t,e^{-t}\rangle$.

The Serret-Frénét formulas, which will give us the three vectors called the Serret-Frénét frame can be deduced in the following way:

$\overrightarrow{t}=\frac{\overrightarrow{v}}{\left|\overrightarrow{v}\right|}$ (tangent vector),

$\overrightarrow{b}=\frac{\overrightarrow{v}\times\overrightarrow{a}}{\left|\overrightarrow{v}\times\overrightarrow{a}\right|}$ (binormal vector), and

$\overrightarrow{n}=\overrightarrow{b}\times\overrightarrow{t}$ (normal vector).

Example: For $t=0$ in the previous curve, we have $\overrightarrow{v}=\langle 0,1,-1\rangle$, and $\overrightarrow{a}=\langle-1,0,1\rangle$, and we compute $\overrightarrow{t}=\left\langle 0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right\rangle$, $\overrightarrow{b}=\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$ and $\overrightarrow{n}=\left\langle\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.

Let us introduce the real valued vector function (or 3 variable scalar function) type $F:V_3\longrightarrow\mathbb{R}$, for which $F(\langle x,y,z\rangle)=F(x,y,z)\in\mathbb{R}$, and the vector valued vector function (shortly vector function) type $\overrightarrow{w}:V_3\longrightarrow V_3$, for which $\overrightarrow{w}(\langle x,y,z\rangle)=\overrightarrow{w}(x,y,z)=\langle F_1(x,y,z),F_2(x,y,z),F_3(x,y,z)\rangle$, where $F_i(x,y,z)$ $(1\leq i\leq 3)$ are of the first type.

We define three operations using a formal vectors, called the vector Nabla, Del operator or Nabla operator, denoted $\overrightarrow{\bigtriangledown}=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle$.

The gradient of a 3 variable scalar function $F:V_3\longrightarrow\mathbb{R}$, for which $F(\langle x,y,z\rangle)=F(x,y,z)\in\mathbb{R}$, is the vector we obtain by mulpilying the vector $\overrightarrow{\bigtriangledown}$ by the scalar F. In other words $\overrightarrow{grad\ F}=F\cdot\overrightarrow{\bigtriangledown}=\left\langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\rangle$.

The divergence of a vector function $\overrightarrow{w}:V_3\longrightarrow V_3$, for which $\overrightarrow{w}(\langle x,y,z\rangle)=\overrightarrow{w}(x,y,z)=\langle F_1(x,y,z),F_2(x,y,z),F_3(x,y,z)\rangle$ by the following "scalar product" $div\ \overrightarrow{w}=\overrightarrow{\bigtriangledown}\cdot\overrightarrow{w}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$.

The curl (rotation) of a vector is $curl\ \overrightarrow{w}=\overrightarrow{\bigtriangledown}\times\overrightarrow{w}=\begin{matrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{matrix}$